On Differential Invariants.[*]

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I. *Introduction.*

This paper is concerned with the expression of certain differential operators and resulting differential parameters in vector form. The chief properties of these parameters are thus shown to be due to the operations involved. It generalizes the Hamiltonian ∇ for space of n dimensions, which may be either flat or curved. From one point of view it has to do with the vector algebra in which the units chosen are themselves variable. The vector algebra used is of the most simple type, the only "product" appearing in the paper being the so-called inner product. A general linear vector operator of one-many type appears and is a vital part of the treatment; however, the formulæ involved are simple. The number of dimensions is general and may in parts of the paper even be infinite.

Memoirs most closely related to this paper are due to Ricci,† Ricci and Levi-Civita,‡ and Maschke§. A paper of Ingold's|| should also be consulted in connection with references to Maschke. The vectors of Ingold's paper will be noticed in the paragraph dealing with the symbolic invariants. The works of Lamé, Christoffel, Beltrami, Bianchi and Darboux should be consulted. Recent memoirs of Bates generalize portions of the theory.

It ought to be evident from a consideration of the results here developed that the method of attack on the problems concerned is natural, free from artifice, and should result in a more penetrating insight into the real nature of these problems.

II. *Vector Algebra of N Dimensions.*¶

1. We shall use the expression called a *vector* in the sense that it represents an ordered set of coordinates, the coordinates being distinguished by the unit attached in each case. Thus we write the vector

$$\rho = x_1 e_1 + \dots + x_n e_n ,$$

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† *Bulletin des Sciences Mathématiques,* (2), 16 (1892), pp. 167–189.

‡ *Mathematische Annalen,* 54 (1900), pp. 125–201.

§ "Present Problems of Algebra and Analysis," *Congress of Arts and Sciences,* St. Louis, 1904, Vol. I, pp. 518–530.

|| *Transactions of the American Mathematical Society,* 11 (1910), pp. 449–474.

¶ Shaw, *Synopsis of Linear Associative Algebra,* pp. 10–15, 32–34.

which enables us to speak of the variables $x_1, \ldots, x_n$, so as to distinguish between them. Two vectors $\rho = \Sigma x_i e_i$ and $\sigma = \Sigma y_i e_i$ are equal if we have the n equations

$$x_i = y_i, \qquad i = 1, \ldots, n.$$

2. We define the expression which may be called the inner product of these two, by the equation

$$I \cdot \rho\sigma = x_1 y_1 + \ldots + x_n y_n .$$

It is obvious that this expression is linear and distributive as to ρ or σ. It is also commutative; that is, $I \cdot \rho\sigma = I \cdot \sigma\rho$. In case $I \cdot \rho\sigma = 0$, we shall call ρ and σ *orthogonal.*

3. We will represent by $A \cdot () A \cdot \alpha\beta$ the linear vector operator $\alpha I \cdot \beta() - \beta I \cdot \alpha()$ or

$$A \cdot () A \cdot \alpha\beta = \begin{vmatrix} \alpha & \beta \\ I \cdot \alpha() & I \cdot \beta() \end{vmatrix},$$

and by $A \cdot \rho A \cdot \alpha\beta$ the expression

$$\begin{vmatrix} \alpha & \beta \\ I \cdot \alpha\rho & I \cdot \beta\rho \end{vmatrix}.$$

It is obvious that this expression is alternating, changes
interchanged, and vanishes if they are equal or if one is a multiple of the other. Likewise, we use a general operator defined by the form

$$A \cdot () \ldots () A \cdot \beta_1 \beta_2 \ldots \beta_m = \begin{vmatrix} \beta_1 & \beta_2 & \ldots & \beta_m \\ I \cdot \beta_1() & I \cdot \beta_2() & \ldots & I \cdot \beta_m() \\ I \cdot \beta_1() & I \cdot \beta_2() & \ldots & I \cdot \beta_m() \\ \ldots & \ldots & \ldots & \ldots \\ I \cdot \beta_1() & I \cdot \beta_2() & \ldots & I \cdot \beta_m() \end{vmatrix},$$

where the different lines of operating symbols act on a set of $m-1$ vectors given in order. We have thus the vector

$$A \cdot \alpha_1 \alpha_2 \ldots \alpha_{m-1} \; A \cdot \beta_1 \beta_2 \ldots \beta_m = \begin{vmatrix} \beta_1 \ldots\ldots\ldots\ldots \beta_m \\ I \cdot \beta_1 \alpha_1 \ldots\ldots I \cdot \beta_m \alpha_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ I \cdot \beta_1 \alpha_{m-1} \ldots I \cdot \beta_m \alpha_{m-1} \end{vmatrix}.$$

It is evident that this vector expression is alternating in the vectors α as well as in the vectors β. It is distributive as to either set. It vanishes if either set are linearly connected among themselves, or if any α is orthogonal to all the vectors β. It is a vector that is linearly expressed in terms of the vectors β. We shall speak of a vector that is linearly expressible in terms of other

linearly independent vectors as co-regional with them. Thus the expression above is co-regional with the vectors β. Let us represent this expression temporarily by σ. Then if we operate on σ by $I\cdot\alpha_i()$, where i runs from 1 to $m-1$, we see that in every case the determinant has two lines alike, and therefore vanishes. That is to say, σ is orthogonal to each of the vectors α. We have here, then, a vector extension of the quaternion form $V\alpha V\beta\gamma$ which is orthogonal to α and in the plane of β and γ.

4. If we operate on σ by the operator $I\cdot\alpha()$ we have

$$I\cdot\alpha A\cdot\alpha_1\alpha_2\ldots\alpha_{m-1}A\cdot\beta_1\beta_2\ldots\beta_m=\begin{vmatrix} I\cdot\alpha\beta_1\ldots\ldots I\cdot\alpha\beta_m \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ I\cdot\alpha_{m-1}\beta_1\ldots I\cdot\alpha_{m-1}\beta_m\end{vmatrix}.$$

This form, however, involves all the vectors $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}$, similarly, and is alternating in all of them, so that we will write it in the form (defining thus the expression)

$$I\cdot A\cdot\alpha\alpha_1\alpha_2\ldots\alpha_{m-1}A\cdot\beta_1\beta_2\ldots\beta_m,$$

and in this form the alternation under the A in either case is more evident. This expression vanishes if either set of vectors, the α's or the β's, are linearly connected among themselves, or if any one of either set is orthogonal to all of the other set. If the vectors reduce to the units e, we have

$$I\cdot A\cdot e_{i_1}\ldots e_{i_m}A\cdot e_{j_1}\ldots e_{j_m}=0 \text{ or } \pm 1$$

according as the subscripts i and j contain members not common to both, or as the two sets are the same subscripts in different order. If the sets agree, the sign is $+$ or $-$ according as the total number of inversions of order in the subscripts is even or odd.

It is obvious that we can expand $A\cdot\alpha_1\alpha_2\ldots\alpha_{m-1}A\cdot\beta_1\beta_2\ldots\beta_m$ in the forms

$$\begin{aligned}&\Sigma\beta_1 I\cdot A\cdot\alpha_1\alpha_2\ldots\alpha_{m-1}A\cdot\beta_2\ldots\beta_m,\\ &\Sigma A\cdot\alpha_1 A\cdot\beta_1\beta_2 I\cdot A\cdot\alpha_2\ldots\alpha_{m-1}A\cdot\beta_3\ldots\beta_m,\\ &\Sigma A\cdot\alpha_1\alpha_2 A\cdot\beta_1\beta_2\beta_3 I\cdot A\cdot\alpha_3\ldots\alpha_{m-1}A\cdot\beta_4\ldots\beta_m,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &\Sigma A\cdot\alpha_1\ldots\alpha_{m-2}A\cdot\beta_1\ldots\beta_{m-1}I\cdot\alpha_{m-1}\beta_m,\end{aligned}$$

where the Σ in each case signifies that the subscripts on the β's are to be permuted in every possible way with a change of sign for every inversion, according to the usual determinant rule of Laplace's expansion.

5. From the form $A\cdot\alpha_1\alpha_2\ldots\alpha_{m-1}A\cdot\beta_1\beta_2\ldots\beta_{m-1}\rho$ we see that we have, by expanding in the first manner above and transposing,

$$\begin{aligned}(-)^m\rho I\cdot A\cdot\alpha_1\ldots\alpha_{m-1}A\cdot\beta_1\beta_2\ldots\beta_{m-1}\\ =\Sigma\beta_1 I\cdot A\cdot\alpha_1\ldots\alpha_{m-1}A\cdot\beta_2\ldots\beta_{m-1}\rho-A\cdot\alpha_1\ldots\alpha_{m-1}A\cdot\beta_1\ldots\beta_{m-1}\rho.\end{aligned}$$

In this form the Σ signifies that the subscripts on the β's are to be permuted with change of sign as above. This formula enables us to express ρ uniquely in terms of two components, one co-regional with the β's, the other orthogonal to all the α's. If the regions of the α's and the β's coincide we have ρ resolved in this region and orthogonal to it. If ρ also lies co-regional with the β's the last term vanishes. If ρ is orthogonal to all the α's, all the terms vanish but the last. These forms are very useful.

6. If we operate on each side of the last equation with $I\cdot\sigma()$, where σ is any vector, we have

$$(-)^m I\cdot\sigma\rho A\cdot\alpha_1\ldots\alpha_{m-1}A\cdot\beta_1\ldots\beta_{m-1}$$
$$=\Sigma I\cdot\sigma\beta_1 IA\cdot\alpha_1\ldots\alpha_{m-1}A\cdot\beta_2\ldots\beta_{m-1}\rho-IA\sigma\alpha_1\ldots\alpha_{m-1}A\beta_1\ldots\beta_{m-1}\rho,$$

from which we derive, since ρ is any vector, the important formula

$$(-)^m\sigma I\cdot A\cdot\beta_1\ldots\beta_{m-1}A\cdot\alpha_1\ldots\alpha_{m-1}$$
$$=\Sigma A\cdot\beta_2\ldots\beta_{m-1}A\cdot\alpha_1\ldots\alpha_{m-1}I\cdot\beta_1\sigma-A\cdot\beta_1\ldots\beta_{m-1}A\cdot\alpha_1\ldots\alpha_{m-1}\sigma.$$

This formula resolves σ into a component co-regional with the α's and a component orthogonal to all the β's. If σ is co-regional with the α's the last term vanishes. If it is orthogonal to all the β's all the terms vanish but the last.

It should be noticed that in these two formulæ the vectors α and β must have a common region of order $m-1$, in order that the coefficient of ρ or σ may not vanish.

7. Let us examine now the linear operator $\phi=\alpha_1 I\beta_1+\alpha_2 I\beta_2+\ldots+\alpha_m I\beta_m$. The transverse of this operator we will designate by ϕ', where $\phi'=\beta_1 I\alpha_1+\beta_2 I\alpha_2+\ldots+\beta_m I\alpha_m$. The *first scalar invariant* of this operator is $m_1=I\alpha_1\beta_1+I\alpha_2\beta_2+\ldots+I\alpha_m\beta_m$. We will define the related operators given below.

$$\chi_1=\phi'-m_1=A\alpha_1 A\beta_1()+A\alpha_1 A\beta_2()+\ldots+A\alpha_m A\beta_m().$$

The *second scalar invariant* of ϕ is then the first invariant of $\phi'\chi_1$ divided by $2!$,

$$m_2=m_1(\phi'\chi_1)/2!=IA\alpha_1\alpha_2 A\beta_2\beta_1+IA\alpha_1\alpha_3 A\beta_3\beta_1+\ldots+IA\alpha_2\alpha_3 A\beta_3\beta_2+\ldots,$$
$$\chi_2=\phi'\chi_1-m_2=A\alpha_1\alpha_2 A\beta_1\beta_2()+A\alpha_1\alpha_3 A\beta_1\beta_3()+\ldots+A\alpha_2\alpha_3 A\beta_2\beta_3()+\ldots.$$

The *third scalar invariant* follows:

$$m_3=m_1(\phi'\chi_2)/3!=IA\alpha_1\alpha_2\alpha_3 A\beta_1\beta_2\beta_3+\ldots,$$
$$\chi_3=\phi'\chi_2-m_3=A\alpha_1\alpha_2\alpha_3 A\beta_1\beta_2\beta_3()+\ldots.$$

The other scalar invariants of ϕ and the operators χ are easily written down in a similar form. Evidently the last one would be the invariant m_m, and the last χ would be χ_m. We may therefore look upon the form $A\alpha_1\ldots\alpha_{m-1}A\beta_1\ldots\beta_m$ as the form χ_{m-1} for $\phi=\alpha_1 I\beta_1+\ldots+\alpha_{m-1}I\beta_{m-1}$, which has then operated upon β_m; that is, $\chi_{m-1}\beta_m$. These forms are useful later.

8. We notice next certain symbolic forms which are abbreviations for longer forms.* If $Q(\alpha, \beta)$ is any expression linear and homogeneous in α and β, then we shall designate by $Q(\zeta, \zeta)$ the expression $\Sigma Q(e_i, e_i)$, where $i=1, 2, \ldots, n$, the number of independent units concerned being n and the units e being orthogonal each to each. This expression will also be defined later in a different manner. Different forms of Q lead to a number of useful formulæ. Thus we have with no difficulty

$$\rho=\zeta I\zeta\rho, \quad I\zeta\zeta=n, \quad A\zeta A\rho\zeta=(n-1)\rho, \quad IA\zeta\lambda A\zeta\mu=I\lambda\mu(n-1),$$
$$A\zeta\lambda_1 A\zeta\mu_1\mu_2=-(n-2)A\lambda_1 A\mu_1\mu_2,$$
$$IA\zeta\lambda_1\ldots\lambda_s A\zeta\mu_1\ldots\mu_s=(n-s)IA\lambda_1\ldots\lambda_s A\mu_1\ldots\mu_s,$$
$$A\zeta\lambda_1\ldots\lambda_s A\zeta\mu_1\ldots\mu_{s+1}=(n-s-1)A\lambda_1\ldots\lambda_s A\mu_1\ldots\mu_{s+1}, \quad \text{etc.}$$

By accenting the symbols ζ we may use several pairs.† Thus

$$I\zeta_1\zeta_2 I\zeta_1\zeta_2=n, \quad IA\zeta_1\zeta_2 A\zeta_1\zeta_2=n(n-1), \quad A\zeta_1\zeta_2 A\rho\zeta_1\zeta_2=(n-2)(n-1)\rho,$$
$$IA\zeta_1\zeta_2\lambda_1 A\zeta_1\zeta_2\mu_1=(n-2)(n-1)I\lambda_1\mu_1,$$
$$A\zeta_1\zeta_2\lambda_1 A\zeta_1\zeta_2\mu_1\mu_2=(n-3)(n-2)A\lambda_1 A\mu_1\mu_2, \quad \text{etc.}$$

In general we have the following reduction formulæ for these symbols:

$$IA\zeta_1\zeta_2\ldots\zeta_s A\zeta_1\zeta_2\ldots\zeta_s=n!/(n-s)!,$$
$$IA\zeta_1\ldots\zeta_t\lambda_1\ldots\lambda_s A\zeta_1\ldots\zeta_t\mu_1\ldots\mu_s=(n-s)!/(n-s-t)!\cdot IA\lambda_1\ldots\lambda_s A\mu_1\ldots\mu_s.$$

If $s+t$ exceeds n these expressions vanish identically. If $s+t=n$ we have only the numerator on the right.

$$A\zeta_1\ldots\zeta_t\lambda_1\ldots\lambda_{s-1}A\zeta_1\ldots\zeta_t\mu_1\ldots\mu_s$$
$$=(-)^t(n-s)!/(n-s-t)!\cdot A\lambda_1\ldots\lambda_{s-1}A\mu_1\ldots\mu_s.$$

III. *Region of Order $N-1$ in Region of Order N.*

1. Let ρ be a vector dependent upon $n-1$ independent parameters, u_1, $u_2, \ldots, u_{n-1}$. That is,

$$\rho=\rho(u_1, \ldots, u_{n-1}).$$

If now we let these parameters vary, and if we assume that the function ρ is differentiable, we have

$$d\rho=\rho_1 du_1+\rho_2 du_2+\ldots+\rho_{n-1}du_{n-1},$$

where ρ_i is the derivative $\partial\rho/\partial u_i$.

2. We shall write as an abbreviation for the orthogonal of $n-1$ vectors

$$(-1)^{i-1}A\alpha_1\alpha_2\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_n Ae_1e_2\ldots e_n/I\cdot A\alpha_1\ldots\alpha_n Ae_1\ldots e_n$$

the vector α_i'. It follows at once that $I\alpha_i\alpha_i'=1$, but $I\alpha_i\alpha_j'=0$ if $i\neq j$.

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* Due to A. McAulay.

† Triple ζ forms, etc., may be used for cubic differential forms, etc.

In case the $n-1$ vectors are $\rho_1, \ldots, \rho_{n-1}$, we shall write ν instead of ρ'_n. The vector ν is thus the normal in the space of n dimensions to the space or region of $n-1$ dimensions. As the parameters are all essential by hypothesis, ν cannot vanish. The vectors $\rho_1, \ldots, \rho_{n-1}$ will be spoken of as tangent to the region. In connection with $\rho_1, \ldots, \rho_{n-1}$ we shall use ρ'_i to represent the orthogonal in the region of $n-1$ dimensions to $n-2$ vectors,

$$\frac{(-1)^{i-1}A\rho_1 \ldots \rho_{i-1}\rho_{i+1} \ldots \rho_{n-1}A\rho_1 \ldots \rho_{n-1}}{I\nu\nu}.$$

We have at once $I\rho_i\rho'_i=1$, $I\rho_i\rho'_j=0$ if $i \neq j$. It is evident that we may expand any vector τ linearly expressible in $\rho_1, \ldots, \rho_{n-1}$ in either of the forms $\tau=\Sigma\rho_i I\rho'_i\tau=\Sigma\rho'_i I\rho_i\tau$.

3. The expressions $I\rho_i\rho_j$, where $i, j=1, 2, \ldots, n-1$, are called the *fundamental quantities of first order.* They may be represented also by a_{ij}. We see that

$$Id\rho d\rho=\Sigma I\rho_i\rho_j du_i du_j, \quad \text{and} \quad I\nu\nu=|a_{ij}|.$$

The expression $Id\rho d\rho$ is called the *first fundamental form.* The partial differentiation of a_{ij} as to u_k gives $I\rho_{ik}\rho_j+I\rho_i\rho_{jk}$. The expression $I\rho_i\rho_{jk}$ is the Christoffel symbol $\begin{bmatrix} jk \\ i \end{bmatrix}$ and the Ricci symbol $a_{jk,i}$. Recalling the significance of ρ'_k we see that the Christoffel symbol $\begin{Bmatrix} ij \\ k \end{Bmatrix}$ is identical with $I\rho_{ij}\rho'_k$. Thence it is easy to see that we have

$$\begin{bmatrix} ij \\ k \end{bmatrix}=\Sigma a_{km}\begin{Bmatrix} ij \\ m \end{Bmatrix}, \qquad m=1, \ldots, n-1.$$

Since $I\rho_i\rho'_i=1$, $I\rho_i\rho'_j=0$, $i \neq j$, we have by differentiation

$$I\rho_{ij}\rho'_k+I\rho_i(\rho'_k)_j=0.$$

Therefore

$$I\rho_{ij}\rho'_k=-I\rho_i(\rho'_k)_j=-I\rho_j(\rho'_k)_i, \quad \text{and} \quad (\rho'_k)_j=-\Sigma\rho'_i I\rho'_k\rho_{ij}.$$

Further we have

$$\begin{aligned}(I\rho_{ij}\rho'_k)_l-(I\rho_{il}\rho'_k)_j &= I\rho_{ij}(\rho'_k)_l-I\rho_{il}(\rho'_k)_j \\ &= \Sigma(I\rho_{il}\rho'_h I\rho_{hj}\rho'_k-I\rho_{ij}\rho'_h I\rho_{hl}\rho'_k), \qquad h=1, \ldots, n-1.\end{aligned}$$

4. Again, if we let $U\nu=\nu/\sqrt{I\nu\nu}$, since $I\rho_i U\nu=0=I\rho_i\nu$, it follows that

$$I\rho_{ij}U\nu=-I\rho_i(U\nu)_j \quad \text{and} \quad I\rho_i\nu_j=-I\rho_{ij}\nu.$$

Because $IU\nu U\nu=1$, therefore $IU\nu(U\nu)_i=0$. Hence $(U\nu)_i$ is wholly in the region of $\rho_1, \ldots, \rho_{n-1}$. Indeed we can write at once $(U\nu)_i=-\Sigma\rho'_m I\rho_{im}U\nu$, *which may also be written in the form of a linear vector function of* ρ_i, namely $N\rho_i$.

The second fundamental form of the region is

$$Id\rho dU\nu=-\Sigma I\rho_{im}U\nu \cdot du_i du_m=Id\rho Nd\rho.$$

The coefficients of this form, $-I\rho_{im}U\nu$, are called the *fundamental quantities of the second order.* They may be represented by D_{im}.

It is easy to see that if we write ν' in place of $A(U\nu)_1\dots(U\nu)_{n-1}Ae_1\dots e_n$, then $I\nu\nu' = |D_{11}, D_{22}, \dots, D_{n-1\,n-1}|$.

Again, if we indicate the partial differentiation of the D's by subscripts,

$$D_{ij,\,m} = -I\rho_{ijm}U\nu - I\rho_{ij}(U\nu)_m, \quad \text{and} \quad D_{im,\,j} = -I\rho_{ijm}U\nu - I\rho_{im}(U\nu)_j,$$

whence

$$\begin{aligned} D_{ij,\,m} - D_{im,\,j} &= I\rho_{im}(U\nu)_j - I\rho_{ij}(U\nu)_m \\ &= \Sigma\,(I\rho_{ij}\rho'_k I\rho_{km}U\nu - I\rho_{im}\rho'_k I\rho_{jk}U\nu) \\ &= \Sigma\left[D_{jk}\begin{Bmatrix} im \\ k \end{Bmatrix} - D_{km}\begin{Bmatrix} ij \\ k \end{Bmatrix}\right]. \end{aligned}$$

These are the extensions of *Codazzi's equations.*

The Christoffel expression $(ijrs)$ is

$$D_{ir}D_{js} - D_{jr}D_{is} = IA\rho_i\rho_j A(U\nu)_r(U\nu)_s = IA\rho_i\rho_j AN\rho_r N\rho_s.$$

Hence the Christoffel quadrilinear covariant G_4 is

$$G_4 = IAd'\rho d''\rho Ad'''U\nu d''''U\nu = \Sigma\,(ijrs)\,du_i du_j du_r du_s.$$

As it is shown later that N is self-transverse, we have at once

$$(ijrs) = (rsij).$$

5. We can write at once a sextilinear covariant as an extension of such forms,

$$\begin{aligned} G_6 &= IAd^{(1)}\rho d^{(2)}\rho d^{(3)}\rho Ad^{(4)}U\nu d^{(5)}U\nu d^{(6)}U\nu \\ &= \Sigma IA\rho_i\rho_j\rho_k AN\rho_r N\rho_s N\rho_t du_i du_j du_k du_r du_s du_t. \end{aligned}$$

The coefficients of this form furnish new symbols. It is obvious that forms may be constructed of this type up to G_{2n-2}, which will consist of one term, whose coefficient is the Kronecker-Gaussian curvature.

6. The differentiation partially of these forms gives new Christoffel symbols, as

$$(ijrst) = \partial(ijrs)/\partial u_t.$$

In the differentiation of the form when written in the vector notation it is to be remembered that N is to be differentiated, differentiation under these conditions being *covariantive differentiation.*

7. The covariant

$$\frac{I\cdot Ad'\rho d''\rho AN d'\rho N d''\rho}{I\cdot Ad'\rho d''\rho Ad'\rho d''\rho}$$

is the *Riemann curvature* for the point and the plane of $d'\rho$, $d''\rho$.

Obvious generalizations would be

$$\frac{IAd'\rho d''\rho d'''\rho ANd'\rho Nd''\rho Nd'''\rho}{IAd'\rho d''\rho d'''\rho Ad'\rho d''\rho d'''\rho}, \quad \text{etc.}$$

We arrive thus again at the Kronecker-Gaussian curvature as the final form.

IV. *The Differential Operator* Δ.

1. We define now the differential operator Δ, which turns out to be of great use,

$$\Delta = \rho_1'\partial/\partial u_1 + \rho_2'\partial/\partial u_2 + \ldots . + \rho_{n-1}'\partial/\partial u_{n-1}.$$

It follows immediately that

$$Id\rho\Delta = du_1\partial/\partial u_1 + \ldots . + du_{n-1}\partial/\partial u_{n-1} = d.$$

That is to say, the operator $Id\rho\Delta$ is a total differentiator, and $I\alpha\Delta$, where α is a unit vector in the region of $\rho_1, \ldots, \rho_{n-1}$, gives the rate of change of the operand in the direction of α. So far as the vector properties of Δ are concerned, it may be treated like any other vector. It is the direct extension of the quaternion ∇. Since it depends upon the derivatives of ρ, the question we have to consider first is whether the parameters in terms of which ρ is expressed can be changed without affecting Δ.

Let the parameters u be expressed in terms of the parameters v by the equations

$$u_i = \phi_i(v_1, v_2, \ldots, v_{n-1}), \qquad i = 1, \ldots, n-1,$$

where it is assumed that the Jacobian

$$J(u) = \frac{\partial(u_1, u_2, \ldots, u_{n-1})}{\partial(v_1, v_2, \ldots, v_{n-1})} \neq 0.$$

Let the differentiation of any expression as to the parameters v be designated by prefixed subscripts, thus, $\partial\rho/\partial v_2 = {}_2\rho$, and let the normal of the region defined by ${}_1\rho, {}_2\rho, \ldots, {}_{n-1}\rho$ be designated by $'\nu$. Then we have

$${}_i\rho = \rho_1\,{}_iu_1 + \rho_2\,{}_iu_2 + \ldots . + \rho_{n-1}\,{}_iu_{n-1}.$$

Hence

$$'\nu = \nu J(u),$$

and therefore

$$U('\nu) = U\nu, \quad \text{and} \quad T('\nu) = T\nu J(u).$$

Further we have (prefixing the $'$ for the v's, instead of affixing it as for the u's)

$$\begin{aligned} {}_i'\rho &= A_1\rho_2\rho \ldots {}_{i-1}\rho_{i+1}\rho \ldots {}_{n-1}\rho A_1\rho_2\rho \ldots {}_{n-1}\rho/I('\nu)('\nu) \\ &= \Sigma\rho_j' \frac{\partial(u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{n-1})}{\partial(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1})}/J(u). \end{aligned}$$

Hence, multiplying into

$$\partial/\partial v_j = {}_ju_1\partial/\partial u_1 + \ldots . + {}_ju_{n-1}\partial/\partial u_{n-1},$$

we have, after summing,

$$'\Delta = \rho_1'\partial/\partial u_1 + \ldots . + \rho_{n-1}'\partial/\partial u_{n-1} = \Delta.$$

We have thus established the invariance of the region as well as the invariance of the operator Δ for the region. If the parameters be looked upon as a set of curvilinear coordinates, the formulæ above enable us to pass from one set of coordinates to the other.

2. It follows that such operators as $I(\,)\Delta\cdot U\nu$, $I(\,)\Delta\cdot\Delta x$, and the like, are also invariants for a transformation of the parameters, as well as their scalar or vector invariants when considered as linear vector operators. In the latter example, $I(\,)\Delta\cdot\Delta x$, an important theorem needs to be proved. Let Φ represent the operator. Then by expanding we have

$$\Phi=\Sigma_j\Sigma_i(\rho_i' x_i)_j I\rho_j'=\Sigma_j\Sigma_i[(\rho_i')_j x_i+\rho_i' x_{ij}]I\rho_j'.$$

The transverse is

$$\breve{\Phi}=\Sigma_j\Sigma_i[\rho_j' I(\rho_i')_j x_i+\rho_j' I\rho_i' x_{ij}].$$

Thus

$$\Phi-\breve{\Phi}=\Sigma_i\Sigma_j[(\rho_i')_j I\rho_j'-\rho_j' I(\rho_i')_j]x_i.$$

If now we operate upon any vector in the region, say ρ_k, where $k=1, 2, \ldots,$ $n-1$, we find that each coefficient for x_i, where $i=1, 2, \ldots, n-1$, vanishes. Hence, *for the region*, Φ is self-transverse; that is, $I\cdot\sigma\Phi\tau=I\cdot\tau\Phi\sigma$.

3. The expression $I\alpha\Delta\cdot Q$ gives the rate of variation of Q in the direction α. It is a linear vector function of α, and in the region of $n-1$ dimensions is self-transverse. There are $n-1$ axes and $n-1$ corresponding roots. In the directions of the axes the rate of variation of Q is itself in the direction α, and its measure is the root corresponding. These axes are the directions of extremal variation; they correspond to the axes of a quadric in the region.

Thus, let us consider the operator $N=I(\,)\Delta\cdot U\nu$. Then we have $Nd\rho$ $=Id\rho\Delta\cdot U\nu$ and $Id'\rho Nd\rho=\Sigma I\rho_i U\nu_j d'u_i du_j=\Sigma I\rho_j U\nu_i d'u_i du_j=Id\rho Nd'\rho$. Hence in the region N is self-transverse. The axes of N are then the directions of extremal rate of change of the unit normal, are orthogonal to each other, and the roots are the curvatures corresponding. One root of N is 0 corresponding to the axis ν. The scalar invariants of N become the mean curvatures, and give us the formulæ

First mean curvature $= I\Delta U\nu$,

Second mean curvature $= \dfrac{1}{2\,!}IA\Delta'\Delta''AU\nu'U\nu''$,

Third mean curvature $= \dfrac{1}{3\,!}IA\Delta'\Delta''\Delta'''AU\nu'U\nu''U\nu'''$,

. . . . ,

$$\text{Total curvature} = \frac{1}{(n-1)\,!}IA\Delta'\ldots\Delta^{(n-1)}AU\nu'\ldots U\nu^{(n-1)}$$
$$= (-1)^{n-1}|D_{11}, \ldots, D_{n-1,n-1}|.$$

The accents here merely mark the proper operand for the differentiator, and are to be removed after the operations are performed. They prevent a covariantive differentiation.

4. The well-known formulæ for the relations between the scalar invariants of a linear vector operator and those of its powers give us formulæ connecting these differential parameters dependent on Φ. Thus the second mean curvature above can be written in the form

$$\frac{1}{2}\begin{vmatrix} I\Delta U\nu, & I\Delta' U\nu'' \\ I\Delta'' U\nu', & I\Delta U\nu \end{vmatrix} = \frac{1}{2}\left[(I\Delta U\nu)^2 - I\Delta' I U\nu' \Delta'' \cdot U\nu''\right].$$

This is the familiar linear operator formula $m_2\Phi = \frac{1}{2}\left[(m_1\Phi)^2 - m_1(\Phi^2)\right]$.

5. Let us consider the operator $I()\Delta \cdot \Delta x$, which gives us the rate of change of the gradient of x in the direction $()$. We have proved that for the region the operator is self-transverse. The scalar invariants are well-known parameters of differential geometry. Thus, using the conventional notation for differential parameters, Δ_1, Δ_2, Δ_{22}, etc.,

$$m_1(\Phi) = \Delta_2 x, \quad m_2(\Phi) = \Delta_{22} x,$$

and we may add others, in a corresponding notation, as,

$$m_3(\Phi) = \Delta_{222} x, \; \ldots\ldots$$

Syzygies connecting these parameters may be deduced from the usual formulæ connecting the scalar invariants. We have thus for $\Phi_x = \Delta I \Delta x() = \Delta x I \Delta()$ the following corresponding parameters,

$$\Delta_1 x = I\Delta x \Delta x, \quad \Delta_1(x, y) = I\Delta x \Delta y.$$

$$\Delta_1(x, \Delta_1 x) = I\Delta x \Delta I \Delta x \Delta x = 2I\Delta x \Phi_x \Delta x.$$

$$\Delta_2(\Delta_1 x) = I\Delta\Delta I\Delta x \Delta x = 2I\Delta\Phi_x\Delta x.$$

In this formula, as elsewhere, the isolated Δ acts upon all following symbols, unless it is accented.

As an example of a well-known formula take the following. Let $n=3$; then we have $\Phi^2 - \Phi \cdot m_1(\Phi) + m_2(\Phi) = 0$. Therefore if we operate upon Δx and take the product $I\Delta x()$, we have, after a transposition and multiplication by 4,

$$4m_2 I\Delta x \Delta x = 4m_1 I\Delta x \Phi \Delta x - 4I\Delta x \Phi^2 \Delta x = 2\Delta_2 x \Delta_1(x, \Delta_1 x) - \Delta_1 \Delta_1 x;$$

that is,

$$4\Delta_{22} \cdot \Delta_1 x = 2\Delta_2 x \Delta_1(x, \Delta_1 x) - \Delta_1 \Delta_1 x.$$

The derivation of this formula by any other method is scarcely as short. Further, the extensions to space of four or more dimensions is obvious. Thus, for four dimensions we have at once

$$\Delta_{222} x \cdot \Delta_1 x = \frac{1}{2}\Delta_{22} x \cdot \Delta_1(x, \Delta_1 x) - \frac{1}{4}\Delta_2 x \cdot \Delta_1(\Delta_1 x) + I\Delta x \Phi^3 \Delta x.$$

The way for further expressions of this type is pointed out here, also an indication that it is better to take as fundamental parameters such forms as

$$I\Delta x\Phi_x^n\Delta x, \quad I\Delta x\Phi_{\Delta_1 x}^n\Delta x,$$

and the like. For example, we have

$$\Phi_{\Delta_1 x}=\Delta I\Delta\Delta_1 x=2\Delta I\Phi_x\Delta x.$$

Thence

$$\Delta_1(x,\,\Delta_1 x)=2I\Delta x\Phi_x\Delta x.$$

$$\Delta_1(\Delta_1 x)=4I\Delta x\Phi_x^2\Delta x.$$

$$\Delta_1(x,\,\Delta_1[x,\,\Delta_1 x])=2I\Delta x\Delta I\Delta x\Phi_x\Delta x=2I\Delta x\Phi_x^2\Delta x+I\Delta x\Phi_{\Delta_1 x}\Delta x.$$

$$\Delta_1(x,\,\Delta_1\Delta_1 x)=4I\Delta x\Phi_{\Delta_1 x}\Phi_x\Delta x.$$

$$\Delta_2\Delta_1 x=m_1(\Phi_{\Delta_1 x}),\ \Delta_{22}(\Delta_1 x)=m_2(\Phi_{\Delta_1 x}),\ \ldots\ldots$$

6. The operator $I(\,)\Delta\cdot U\Delta x=\Phi'_x$ gives the rate of variation of the unit normal of the region of order $n-2$ determined by the levels of the function x, in the region of order $n-1$, and is thus the extension of the geodetic curvature. Its roots and axes determine the extremals of these. The invariants $I\Delta U\Delta x$, etc., give what may be called geodetic mean curvatures. If we write $T\Delta x$ for $\sqrt{I\Delta x\Delta x}$ we have

$$\Phi'_x=\frac{\Phi_x}{T\Delta x}-\frac{1}{2}\Delta x\frac{I\Delta x\Phi_x}{T^3\Delta x}.$$

The first geodetic curvature reduces to

$$I\Delta\left(\frac{\Delta x}{T\Delta x}\right)=\frac{I\Delta\Delta x}{T\Delta x}-\frac{1}{2}\frac{I\Delta x\Delta I\Delta x\Delta x}{T^3\Delta x}=\frac{\Delta_2 x}{\sqrt{\Delta_1 x}}+\Delta_1\left(x,\,\frac{1}{\sqrt{\Delta_1 x}}\right).$$

This formula is well-known. There will be many other functions related to Φ'_x, and others analogous to all these so far considered, but we need not insist upon them here.

V. *Symbolic Invariants.*

1. We return now to the previous double ζ notation. By the use of this we find expressions for all the forms used by Prof. Maschke.* The following examples of Maschke's forms indicate the equivalences. (We abbreviate thus: for the scalar $I\cdot A\alpha_1\alpha_2\ldots\alpha_n Ae_1\ldots e_n$ we use $A\cdot\alpha_1\alpha_2\ldots\alpha_n$. It is clear that if α is linear in $\rho_1,\rho_2,\ldots,\rho_{n-1}$, then $A\cdot\alpha_1\ldots\alpha_{n-1}\nu=IA\alpha_1\ldots\alpha_{n-1}A\rho_1\ldots\rho_{n-1}$.)

$$f_i=I\zeta\rho_i,\quad a_{ik}=I\rho_i\rho_k,\quad f_{kl}=I\zeta\rho_{kl}.$$

$$(f)=A\zeta_1\zeta_2\ldots\zeta_{n-1}U\nu,\quad (f)^2=(n-1)\,!\,IU\nu U\nu=(n-1)\,!\,.$$

$$(uf)=A\Delta u\zeta_1\ldots\zeta_{n-2}U\nu,$$

$$(uf)^2=(n-2)\,!\,IA\Delta uU\nu A\Delta uU\nu=(n-2)\,!\,I\Delta u\Delta u=(n-2)\,!\,\Delta_1 u.$$

* Present Problem of Algebra and Analysis, *Congress of Arts and Sciences*, St. Louis, 1904, Vol. I, pp. 518–530; *Transactions of the American Mathematical Society*, Vols. I, VII.

$$(uf)(vf)=(n-2)!\,\Delta_1(u,v),\quad (uvf)=A\Delta u\Delta v\zeta_1\ldots\zeta_{n-3}U\nu,$$

$$(uvf)^2=(n-3)!\,IA\Delta u\Delta vU\nu A\Delta u\Delta vU\nu=(n-3)!\,IA\Delta u\Delta vA\Delta u\Delta v.$$

$$((uf),f)=A(\Delta A\Delta u\zeta_1\ldots\zeta_{n-2}U\nu)\zeta_1\ldots\zeta_{n-2}U\nu$$
$$=(n-2)!\,IAU\nu\Delta AU\nu\Delta u=(n-2)!\,\Delta_2 u.$$

$$(fa)(fv)(ua)=A\zeta_1\eta_1\ldots\eta_{n-2}U\nu A\zeta_1\Delta v_1\ldots\Delta v_{n-2}U\nu A\Delta u\eta_1\ldots\eta_{n-2}U\nu$$
$$=A\Delta u\Delta v_1\ldots\Delta v_{n-2}U\nu\cdot(n-2)!.$$

When we need to discriminate between sets of ζ's we accent them or use η.

2. It is obvious that if we differentiate $(f)^2$ as to u_i we have at once

$$(f)(f)_i=0.$$

Such differentiation may be extended to any of these forms, remembering that the Δ implicit in them is also differentiable (the covariantive differentiation of Maschke), as for example,

$$(fa)_i=A\zeta\eta_1\ldots\eta_{n-2}U\nu_i,$$

and

$$(uf)_i=A\Delta u_i\zeta_1\ldots\zeta_{n-2}U\nu+A\Delta_i u\zeta_1\ldots\zeta_{n-2}U\nu+A\Delta u\zeta_1\ldots\zeta_{n-2}U\nu_i.$$

Thus we have such formulæ as

$$f_k(fa)_i(ua)=0,$$
$$(fa)_i(fv)(ua)=0.$$

3. The Maschke symbol $[f]=A\cdot\zeta_1\zeta_2\ldots\zeta_{n-1}z$, where z may be $U\nu$, or $(U\nu)_i$, etc. Thus we have

$$f_1'(uf)[f]=(n-2)!\,u_1IU\nu z,$$
$$f_i(ua)[fa]=(n-2)!\,I\Delta aq_iIU\nu z,$$
$$f_k(fa)_i(ua)=0,$$
$$(fa)(fv)(ua)=A\Delta u\Delta v_1\ldots\Delta v_{n-2}U\nu\cdot(n-2)!,$$
$$(fa)_i(fv)(ua)=0,$$
$$f_1'f_2''(uvf)[f]=(n-3)!\,IU\nu zIA\Delta u\Delta vA\rho_i\rho_j.$$

4. It is not necessary to point out further the equivalent forms, as they can be easily written down and reduced from the system of Maschke to that of the present paper. The forms given here are at once interpretable in geometric form and for that reason would seem to be more useful in the end. The vectors of Ingold's paper, cited above, are the vectors (other than ζ) appearing above, and his formulæ are not the above equivalents of Maschke's but are their analogues due to suppressing the ζ's and the scalar function.

Applications of these operational forms to differential geometry or mechanics may easily be found in the theorems of the treatises and memoirs cited.